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Sweeping process with regular and nonregular sets

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Abstract

Differential inclusions involving the normal cone to a moving set are investigated. A special attention is paid to the sweeping process associated with sets for which no regularity assumption is required.

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1. Introduction

Let T be a positive real number and let, for each $t \in [0, T]$, a nonempty closed subset $C(t)$ of a Hilbert space H. We will be concerned, for any fixed $x_0 \in C(0)$, with the differential inclusion

$$\text{(I)} \qquad \dot{x}(t) \in -N_{C(t)}(x(t)), \quad x(0) = x_0 \in C(0),$$

where for any subset S in H the set $N_S(u)$ denotes the Clarke normal cone to S at $u \in S$. Sometimes, we will write $N(S; u)$ in place of $N_S(u)$. By a solution of (I), one generally means an absolutely continuous mapping x from $[0, T]$ into H such that

$$x(0) = x_0 \quad \text{and} \quad \dot{x}(t) \in -N_{C(t)}(x(t)) \quad \text{for almost all } t \in [0, T].$$

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The differential inclusion (I) is generally called the sweeping process and it has been introduced and thoroughly studied in the 70s by Moreau [23–26] in the setting where all the sets $C(t)$ are assumed to be convex. Then Castaing [4] studied the stochastic version in the same setting. Castaing also introduced in [5] some new techniques from which one can derive several results, in particular the existence of a solution of (I) when $C(t)$ is the form $C(t) = S + v(t)$, where S is any fixed closed subset (with H finite dimensional) and v has a finite variation. Considering the smallest set-valued mapping G whose graph is closed and contains the graph of $(t,u) \mapsto N_{C(t)}(u) \cap \mathbb{B}_H$ (for C 1-Lipschitz), Valadier [32] showed that the differential inclusion

$$\dot{x}(t) \in -G(t, x(t)), \quad x(0) = x_0 \in C(0)$$

admits at least a solution. So, he obtained a solution for (I) whenever the set-valued mapping $(t,u) \mapsto N_{C(t)}(u) \cap \mathbb{B}_H$ has a closed graph, providing in this way a solution to (I) for the new case where $C(t)$ is the complement of the interior of a convex set. See also [18], [21], [33] for some other contributions. Recently Benabdellah [2] and Colombo–Goncharov [13] independently showed the existence of a solution of (I) when the sets $C(t)$ are general nonconvex closed sets (with H finite dimensional) moving in a Lipschitzian way with respect to t.

In the same 70s period, Henry [19] introduced for the study of planning procedures in mathematical economy the differential inclusion

$$\dot{x}(t) \in \mathrm{Proj}_{T(K;x(t))}(F(x(t)), \quad x(0) = x_0 \in K,$$

where F is an upper semicontinuous set-valued mapping with nonempty compact convex values, K is a closed convex set, and $T(K;.)$ denotes the tangent cone to K. Some years later, the same problem has been studied by Cornet [14,15] who assumed only the Clarke regularity of the set K instead of its convexity. Cornet and Henry reduced in some way the problem to the existence of a solution of the differential inclusion

$$(\mathrm{I}_1) \qquad \dot{x}(t) \in -N_K(x(t)) + F(x(t)), \quad x(0) = x_0 \in K.$$

In fact, Cornet [14] thoroughly studied the relationship between the two differential inclusions. We also refer to Malivert [20] for other important classes of sets K.

Perturbations of the differential inclusion (I) in the form

$$(\mathrm{I}_2) \qquad \dot{x}(t) \in -N_{C(t)}(x(t)) + \Gamma(t, x(t))$$

with $C(t)$ convex or the complement of the interior of a convex set have been considered by Castaing–Duc Ha–Valadier [6], Castaing–Monteiro Marques [7] and several authors (see, e.g. the references in [6,22]). Corresponding functional (with delay) perturbed differential inclusions (see Section 3 for the notation x_t)

$$(\mathrm{I}_3) \qquad \dot{x}(t) \in -N_{C(t)}(x(t)) + \Phi(t, x_t)$$

(denoted by (III) in Section 3) also appear in Castaing–Monteiro Marques [8] with the assumption of convexity of $C(t)$ or of its complement.

Our main purpose in this paper is to show how a classical differential inclusion with convex compact values is strongly connected with the differential inclusions (I), (I_1), (I_2) and (I_3). It allows us to obtain the existence of a solution to the differential inclusion (I) in the case when the sets $C(t)$ are nonconvex and move in an absolutely continuous way. Indeed, assuming

$$|d(x, C(t)) - d(x, C(s))| \leqslant |v(t) - v(s)| \quad \text{for all } x \in H, \ s, t \in [0, T],$$

where v is a nondecreasing continuous function, we prove that the differential inclusion

$$\text{(II)} \qquad \dot{x}(t) \in -\dot{v}(t)\partial d_{C(t)}(x(t)), \quad x(0) \in C(0)$$

(where $\partial d_S(u)$ denotes the Clarke subdifferential of the distance function d_S) admits at least one absolutely continuous solution satisfying the constraint $x(t) \in C(t)$ for all $t \in [0, T]$. In so doing, such a solution of (II) provides a solution of (I) under the same general assumption. Our method strongly uses a viability theorem by Frankowska and Plaskacz [16] for differential inclusions over tubes. This is developed in Section 4.

The connection of (II) with the differential inclusion (I) is reinforced by the fact that we establish in Section 2 that a mapping is a solution of (I) if and only if it is a solution of (II) when the nonconvex sets $C(t)$ are regular in some sense. Our methods can be adapted to provide in the third section the existence of a solution of the perturbed differential inclusions (I_1) and (I_2) via the differential inclusion

$$\dot{x}(t) \in -\dot{w}(t)\partial d_{C(t)}(x(t)) + \Gamma(t, x(t)),$$

where w is an absolutely continuous function associated with the function v above and with the growth condition of the set-valued mapping Γ. A similar result is also proved in the case of the functional perturbed differential inclusion (I_3). This makes clear another contribution of our paper providing in a unified way the existence of solutions of the sweeping process and the differential inclusion (I_1) associated with the Henry differential inclusion above. At the same time, we provide new insights on the differential inclusion (I_1) in establishing the existence of solutions for any nonconvex closed set K.

2. Sweeping process with regular sets

Recalling that the Clarke normal cone $N_S(u)$ (see [10,11]) is empty at any point $u \notin S$, one observes that the differential inclusion (I) is implicitly subject to the constraints

$$x(t) \in C(t) \quad \text{for all } t \in [0, T],$$

whenever the graph of C is closed (this will be the case below).

In all the paper, we will often assume that the closed sets $C(t)$ satisfy the following hypothesis: there exists a nondecreasing absolutely continuous function v from $[0, T]$

into $\mathbb{R}$ such that for all $s, t \in [0, T]$ and all $e \in H$

$$(\mathscr{H}) \qquad |d(e, C(t)) - d(e, C(s))| \leqslant |v(t) - v(s)|.$$

This amounts to saying that C moves in an absolutely continuous way with respect to the Hausdorff distance. Sometimes, we will require the weaker hypothesis

$$(\mathscr{H}') \qquad d(e, C(t)) - d(e, C(s)) \leqslant v(t) - v(s),$$

for all $e \in H$ and $s, t \in [0, T]$ with $s \leqslant t$. This means that C has an absolutely continuous retraction (see [25]).

Recall that $x^* \in X^*$ is in the *Fréchet subdifferential* $\partial^F f(x)$ of a function f from a normed space X into $\mathbb{R} \cup \{+\infty\}$ with $f(x) < \infty$ provided for each $\varepsilon > 0$ there exists some neighborhood U of x such that for all $u \in U$ one has

$$\langle x^*, u - x \rangle \leqslant f(u) - f(x) + \varepsilon ||u - x||.$$

When f is the indicator function ψ_S of a subset $S \subset X$ and $x \in S$ (recall that $\psi_S(x) = 0$ if $x \in S$ and $\psi_S(x) = +\infty$ if $x \notin S$), this amounts to saying that for some neighborhood U of x one has for all $u \in U \cap S$

$$\langle x^*, u - x \rangle \leqslant \varepsilon ||u - x||.$$

The obtained set is called the *Fréchet normal cone* to S at x and it is denoted by $N_S^F(x)$.

When $\partial^F g(x)$ coincides with the Clarke subdifferential $\partial f(x)$ of f at x, one says that f is *subdifferentially regular* at x. The regularity of the indicator function ψ_S is equivalent to the equality $N_S^F(x) = N_S(x)$, i.e., the Clarke and the Fréchet normal cones to S at $x \in S$ coincide. Generally, one says that the set S is *normally regular* at $x \in S$. One knows by Bounkhel and Thibault [3] that, in the setting of Hilbert space, this is equivalent to the subdifferential regularity at $x \in S$ of the distance function $d_S := d(., S)$. When S is normally regular at all points in S, we merely say that S is normally regular.

We first prove that, for regular sets, any solution of (I) is a solution of a differential inclusion associated with the distance function $d_{C(t)}(.)$.

Proposition 2.1. *Assume that* $(\mathscr{H}')$ *holds. If* $x(.)$ *is a solution of* (I) *and if all the sets* $C(t)$ *are normally regular, then* $x(.)$ *is also an absolutely continuous solution of the differential inclusion*

$$\text{(II)} \qquad \dot{x}(t) \in -\dot{v}(t)\partial d_{C(t)}(x(t)), \quad x(0) = x_0 \in C(0).$$

Proof. Fix any $t \in]0, T[$ with x and v derivable at t and with $\dot{x}(t) \neq 0$. Then, by definition of (I) and the remark above, one has $x(t) \in C(t)$ and by the regularity of $C(t)$ one has

$$-\frac{\dot{x}(t)}{||\dot{x}(t)||}\in N^F_{C(t)}(x(t)).$$

But (see, e.g. [3] and the references therein) one knows that for any closed set S and $u\in S$

$$\partial^F d_S(u) = N^F_S(u)\cap\{u^*\in H : ||u^*||\leqslant 1\}.$$

Therefore, one has

$$-\frac{\dot{x}(t)}{||\dot{x}(t)||}\in \partial^F d_{C(t)}(x(t)). \tag{2.1}$$

Fix any $\varepsilon>0$. For $s<t$ sufficiently close to t we can write

$$\begin{aligned}\left\langle -\frac{\dot{x}(t)}{||\dot{x}(t)||}, x(s)-x(t)\right\rangle &\leqslant d_{C(t)}(x(s)) + \varepsilon||x(s)-x(t)||\\ &= d_{C(t)}(x(s)) - d_{C(s)}(x(s)) + \varepsilon||x(s)-x(t)||\\ &\leqslant v(t)-v(s)+\varepsilon||x(s)-x(t)||\end{aligned}$$

and hence

$$\left\langle -\frac{\dot{x}(t)}{||\dot{x}(t)||}, \frac{x(s)-x(t)}{t-s}\right\rangle \leqslant \frac{1}{t-s}(v(t)-v(s)) + \varepsilon.\frac{1}{t-s}||x(s)-x(t)||,$$

that is,

$$\left\langle \frac{\dot{x}(t)}{||\dot{x}(t)||}, \frac{x(s)-x(t)}{s-t}\right\rangle \leqslant \frac{v(s)-v(t)}{s-t} + \varepsilon\left|\left|\frac{x(s)-x(t)}{s-t}\right|\right|.$$

Taking the limits for $s\to t$ with $s<t$ we get

$$\left\langle \frac{\dot{x}(t)}{||\dot{x}(t)||}, \dot{x}(t)\right\rangle \leqslant \dot{v}(t) + \varepsilon||\dot{x}(t)||,$$

that is, $||\dot{x}(t)||\leqslant \dot{v}(t)+\varepsilon||\dot{x}(t)||$ and hence $||\dot{x}(t)||\leqslant\dot{v}(t)$. As the inequality obviously holds when $\dot{x}(t)=0$, we obtain $||\dot{x}(t)||\leqslant\dot{v}(t)$ for almost all $t\in[0,T]$ and hence by the definition of the Fréchet subdifferential and by (2.1)

$$-\dot{x}(t)\in\dot{v}(t)\partial^F d_{C(t)}(x(t)).$$

As the Fréchet subdifferential is always included in the Clarke one, the last relation above allows us to conclude that

$$-\dot{x}(t)\in\dot{v}(t)\partial d_{C(t)}(x(t)). \qquad \square$$

Consider also the case when the sets $C(t)$ are complements of normally regular sets. Recall first that a subset $S \subset H$ is epi-Lipschitzian (see Rockafellar [29]) at a point $\bar{x} \in S$ if there are $r > 0$ and a vector $y \in H$ such that

$$(\bar{x} + r\mathbb{B}_H) \cap S + [0, r](y + r\mathbb{B}_H) \subset S.$$

(Here $\mathbb{B}_H$ denotes the closed unit ball of H). Such a set S obviously has a nonempty interior. Rockafellar [29] proved, for the Clarke tangent cone $\mathrm{Tang}(S; \bar{x})$, the equality

$$\mathrm{Tang}(H \setminus \operatorname{int} S; \bar{x}) = -\mathrm{Tang}(S; \bar{x}),$$

whenever S is epi-Lipschitzian at $\bar{x}$ in the boundary bd S of S, and hence by polarity

$$N(H \setminus \operatorname{int} S; \bar{x}) = -N(S; \bar{x}). \tag{2.2}$$

When S is epi-Lipschitzian at all points in S, one says that S is epi-Lipschitzian. Any closed convex set with nonempty interior is epi-Lipschitzian. The statement of the proposition below also uses the set-valued mapping E_S associated with a subset of the form $S = H \setminus \operatorname{int} S'$ and defined by

$$E_S(x) = \partial d_{S'}(x) \text{ if } x \in \operatorname{bd} S, \quad E_S(x) = \{0\} \text{ if } x \in \operatorname{int} S, \quad \text{and}$$

$$E_S(x) = \emptyset \text{ if } x \notin S.$$

Similar results of equivalence have been first proved in different way by Valadier [32] when the sets $C'(t)$ below are convex sets with nonempty interior. Note that the control result, as (2.3), of $\|\dot{x}(t)\|$ in [32] is provided for any solution x of the differential inclusion

$$\dot{x}(t) \in N_{C'(t)}(x(t)), \quad x(0) = x_0 \notin \operatorname{int} C'(0)$$

with the constraints $x(t) \notin \operatorname{int} C'(t)$ for all t, where $C'(t)$ are convex sets (with maybe empty interiors) satisfying a condition in the line of $(\mathscr{H})$.

Proposition 2.2. *Let $C'(t)$ be closed subsets of H that are normally regular and epi-Lipschitzian and denote by $C(t)$ the complement of* int $C'(t)$, *i.e.*, $C(t) := H \setminus \operatorname{int} C'(t)$. *Instead of assuming $(\mathscr{H}')$ for $C(t)$, assume that there exists a nonincreasing absolutely continuous function v such that*

$$d(e, C'(t_1)) - d(e, C'(t_2)) \leqslant v(t_1) - v(t_2)$$

for all $e \in H$, $t_1, t_2 \in [0, T]$ with $t_1 \leqslant t_2$. Then, $x(.)$ is a solution of (I) *if and only if it is a solution of the differential inclusion*

$$\dot{x}(t) \in |\dot{v}(t)| E_{C(t)}(x(t)) \quad \textit{and} \quad x(0) = x_0 \in C(0).$$

In particular, for any solution $x(.)$ *of* (I) *one has*

$$||\dot{x}(t)|| \leqslant |\dot{v}(t)| \quad \textit{for almost all } t \in [0, T]. \tag{2.3}$$

Proof. Suppose first that $x(.)$ is a solution of the differential inclusion in the statement of the proposition and let $t \in [0, T]$ where the inclusion holds. Then $x(t) \in C(t)$ and if $x(t) \in \operatorname{int} C(t)$, we have $\dot{x}(t) = 0$ because of the definition of $E_{C(t)}$ and hence $\dot{x}(t) \in -N_{C(t)}(x(t))$. If $x(t) \in \operatorname{bd} C(t)$, then $x(t) \in \operatorname{bd} C'(t)$ because

$$\operatorname{bd} C(t) = \operatorname{bd}(\operatorname{int} C'(t)) = \operatorname{cl}(\operatorname{int} C'(t)) \setminus \operatorname{int} C'(t) \subset \operatorname{bd} C'(t).$$

So, we have

$$x(t) \in \operatorname{bd} C'(t) \quad \text{and} \quad \dot{x}(t) \in |\dot{v}(t)| \partial d_{C'(t)}(x(t)) \subset N_{C'(t)}(x(t)).$$

According to (2.2), we obtain $\dot{x}(t) \in -N_{C(t)}(x(t))$. Thus, $x(.)$ is a solution of (I).

Suppose now that $x(.)$ is a solution of (I) and fix $t \in]0, T[$ where the inclusion of (I) holds. We have $x(t) \in C(t)$ and, as $N_S(u) = \{0\}$ whenever $u \int S$, it is enough to consider the case $x(t) \in \operatorname{bd} C(t)$. Suppose further that $\dot{x}(t) \neq 0$. Then, according to (2.2) we have

$$\frac{\dot{x}(t)}{||\dot{x}(t)||} \in -N_{C(t)}(x(t)) = N_{C'(t)}(x(t))$$

and (as in the proof of Proposition 2.1) the normal regularity of $C'(t)$ entails

$$\frac{\dot{x}(t)}{||\dot{x}(t)||} \in \partial^F d_{C'(t)}(x(t)). \tag{2.4}$$

Fix any $\varepsilon > 0$. For $s > t$ sufficiently close to t, we obtain (as in Proposition 2.1)

$$\left\langle \frac{\dot{x}(t)}{||x(t)||}, x(s) - s(t) \right\rangle \leqslant |v(s) - v(t)| + \varepsilon ||x(s) - x(t)||.$$

Dividing by $s - t > 0$ and taking the limit with $s \downarrow t$ yields

$$\left\langle \frac{\dot{x}(t)}{||\dot{x}(t)||}, \dot{x}(t) \right\rangle \leqslant |\dot{v}(t)| + \varepsilon ||\dot{x}(t)||$$

and hence $||\dot{x}(t)|| \leqslant |\dot{v}(t)|$. Using this inequality in (2.4) gives (taking the definition of the Fréchet subdifferential into account)

$$\dot{x}(t) \in ||\dot{x}(t)|| \partial^F d_{C'(t)}(x(t)) \subset |\dot{v}(t)| \partial^F d_{C'(t)}(x(t)) \subset |\dot{v}(t)| \partial d_{C'(t)}(x(t))$$

and hence $\dot{x}(t) \in |\dot{v}(t)| E_{C(t)}(x(t))$. The proof is then complete. $\square$

We proceed now to proving the converse of Proposition 2.1 when the distance function $d_{C(t)}(.)$ is subdifferentially regular at all points near the set $C(t)$. So, we assume that there exists some number $\rho(t)>0$ such that $d(.)$ is subdifferentially regular at any point in the enlargement $C(t)+\rho(t)\mathbb{U}_H$. Here $\mathbb{U}_H$ denotes the open unit ball of H. It follows from Poliquin et al. [27] (see Theorem 4.1 therein) that this amounts to requiring that the sets $C(t)$ are (uniformly) $\rho(t)$-prox-regular with respect to the open tube $\{u\in H : 0<d_{C(t)}(u)<\rho(t)\}$. Such sets will be called $\rho(t)$-*prox-regular* in the paper. They are called $\rho(t)$-proximally smooth in Clarke et al. [12]. Note that any point in $C(t)+\rho(t)\mathbb{U}_H$ (see [12,27]) has one and only one projection over $C(t)$.

We also assume that

$$2\int_0^t \dot{v}(s)\,ds<\rho(t) \quad \text{for } t>0. \tag{2.5}$$

Lemma 2.1. *Under* $(\mathscr{H}')$ *and* (2.5), *for any solution* $x(.)$ *of* (II) *one has*

$$x(t)\in C(t)+\rho(t)\mathbb{U}_H \quad \textit{for any } t\in[0,T].$$

Proof. Fix any solution x of (II). Then, inclusion (II) ensures that $\|\dot{x}(t)\|\leqslant\dot{v}(t)$ for almost all $t\in[0,T]$ and hence

$$\begin{aligned} d(x(t),C(t)) &\leqslant \|x(t)-x(0)\|+d(x(0),C(t)) \\ &= \|x(t)-x(0)\|+d(x(0),C(t))-d(x(0),C(0)) \\ &\leqslant \int_0^t \|\dot{x}(s)\|\,ds+v(t)-v(0) \\ &\leqslant 2\int_0^t \dot{v}(s)\,ds. \end{aligned}$$

This implies that

$$d(x(t),C(t))<\rho(t). \qquad \square$$

Using the techniques developed in Thibault [30] to get existence of solutions for sweeping process with convex sets, we can prove the following theorem. It establishes coincidence between the solution sets of (I) and (II) under prox-regularity assumption.

Theorem 2.1. *Assume that* $C(t)$ *is* $\rho(t)$*-prox-regular and that* $(\mathscr{H})$ *and* (2.5) *hold. A mapping* $x(.)$ *is a solution of the constrained differential inclusion* (I) *iff it is a solution of the unconstrained differential inclusion* (II).

Proof. Taking Proposition 2.1 into account, it is sufficient to prove that any solution of (II) is a solution of (I). So, fix a solution $x(.)$ of (II). It is enough to prove that $x(t) \in C(t)$ for all $t \in [0, T]$. Put $h(t) = d(x(t), C(t))$. It follows from $(\mathscr{H})$ that h is absolutely continuous. The set

$$\Omega = \{t \in [0, T] : x(t) \notin C(t)\}$$

is then open in $[0, T]$ because $\Omega = \{t \in [0, T] : h(t) > 0\}$. We claim that this set Ω is in fact empty. Indeed, let us suppose that $\Omega \neq \emptyset$. Then, as $0 \notin \Omega$, there exists a nonempty open interval $]\alpha, \beta[\subset \Omega$ such that $h(\alpha) = 0$ (it suffices to choose for $]\alpha, \beta[$ any connected component of $\Omega \cap]0, T[\neq \emptyset$). Let s be any point of $]\alpha, \beta[$ where $\dot{h}(s)$, $\dot{x}(s)$ and $\dot{v}(s)$ exist and such that $\dot{x}(s)$ satisfies the inclusion (II). Then for such a point s we have for $\delta > 0$ sufficiently small and for some mapping $\varepsilon(\delta) \xrightarrow[\delta \downarrow 0]{} 0$

$$\begin{aligned}
&\delta^{-1}[h(s+\delta) - h(s)] \\
&\quad = \delta^{-1}[d(x(s+\delta), C(s+\delta)) - d(x(s), C(s))] \\
&\quad = \delta^{-1}[d(x(s) + \delta\dot{x}(s) + \delta\varepsilon(\delta), C(s+\delta)) - d(x(s) + \delta\dot{x}(s), C(s+\delta))] \\
&\qquad + \delta^{-1}[d(x(s) + \delta\dot{x}(s), C(s+\delta)) - d(x(s) + \delta\dot{x}(s), C(s))] \\
&\qquad + \delta^{-1}[d(x(s) + \delta\dot{x}(s), C(s)) - d(x(s), C(s))] \\
&\quad \leqslant ||\varepsilon(\delta)|| + \delta^{-1}(v(s+\delta) - v(s)) \\
&\qquad + \delta^{-1}[d(x(s) + \delta\dot{x}(s), C(s)) - d(x(s), C(s))].
\end{aligned}$$

If $\dot{v}(s) = 0$, then (II) says that $\dot{x}(s) = 0$ and hence we get from the last inequality

$$\dot{h}(s) \leqslant \dot{v}(s) = 0.$$

Suppose now $\dot{v}(s) \neq 0$. As $x(s) \notin C(s)$, we observe by the lemma above and Theorem 4.1 in Poliquin et al. [27] that the function $d(., C(s))$ is Fréchet differentiable at $x(s)$. So, inclusion (II) for $\dot{x}(s)$ means that $-\frac{\dot{x}(s)}{\dot{v}(s)}$ is the Fréchet differential at $x(s)$ of the function $d(., C(s))$. This implies that $||\dot{x}(s)|| = \dot{v}(s)$ and for some function $\eta(\delta) \to 0$

$$\begin{aligned}
\delta^{-1}[d(x(s) + \delta\dot{x}(s), C(s)) - d(x(s), C(s))] &= \left\langle -\frac{\dot{x}(s)}{\dot{v}(s)}, \dot{x}(s) \right\rangle + \eta(\delta) \\
&= -\dot{v}(s) + \eta(\delta).
\end{aligned}$$

This equality and the inequality above concerning $\delta^{-1}[h(s+\delta) - h(s)]$ entail that one has

$$\dot{h}(s) \leqslant \dot{v}(s) - \dot{v}(s) \leqslant 0.$$

Therefore, for almost all $s \in]\alpha, \beta[$ we have $\dot{h}(s) \leqslant 0$ and hence for every $t \in]\alpha, \beta[$ we obtain

$$h(t) = h(\alpha) + \int_\alpha^t \dot{h}(s)\, ds = \int_\alpha^t \dot{h}(s)\, ds \leqslant 0,$$

which is a contradiction with $]\alpha, \beta[\subset \Omega$. So, $\Omega = \emptyset$ and the proof is complete. $\square$

The existence theorem for sweeping process with prox-regular sets can now be derived.

Theorem 2.2. *Assume that H is finite dimensional and $C(t)$ is $\rho(t)$-prox-regular. Assume also that $(\mathscr{H})$ and* (2.5) *hold. Then, the differential inclusions* (I) *and* (II) *have the same solution set and this solution set is nonempty. Further, for any solution $x(.)$ one has $\|\dot{x}(t)\| \leqslant \dot{v}(t)$ for almost all $t \in [0, T]$.*

Proof. By what precedes, it is enough to show that the unconstrained differential inclusion (II) has a solution. Put

$$f_t(x) := f(t, x) := -\dot{v}(t) d_{C(t)}(x).$$

It is easy to verify that the set-valued mapping $(t, x) \mapsto \partial f_t(x)$ has nonempty convex compact values, is measurable with respect to (t, x) and upper semicontinuous with respect to x. It is also easily seen that

$$\partial f_t(x) \subset \dot{v}(t) \mathbb{B}_H.$$

Then, it follows (see, e.g., [9, Theorem VI-13]) that the differential inclusion (II) has at least one solution. $\square$

Consider now a general case with uniqueness. We put the convention $1/\infty = 0$.

Corollary 2.1. *In addition to the assumptions of Theorem* 2.2, *assume that the function $t \mapsto \dot{v}(t)/\rho(t)$ is integrable over $[0, T]$. Then, the sweeping process* (I) *admits one and only one solution $x(.)$, and this solution $x(.)$ satisfies $\|\dot{x}(t)\| \leqslant \dot{v}(t)$. Further, if $\mathscr{S}_a$ denotes the unique solution associated with $a \in C(0)$, then the mapping $a \mapsto \mathscr{S}_a$ is Lipschitz from $C(0)$ into the space of continuous mappings from $[0, T]$ into H endowed with the sup norm.*

Proof. The existence result follows from Theorem 2.2. Consider now two points a, b in $C(0)$. Let x^a (resp. x^b) be a solution associated with the initial value a (resp. b).

By Theorem 2.2, we have for almost all $t \in [0, T]$

$$-\dot{x}^a(t) \in \dot{v}(t)\partial d_{C(t)}(x^a(t)) \quad \text{and} \quad -\dot{x}^b(t) \in \dot{v}(t)\partial d_{C(t)}(x^b(t)),$$

which ensures by Theorem 4.1. in Poliquin et al. [27] that

$$\langle \dot{x}^b(t) - \dot{x}^a(t), x^a(t) - x^b(t) \rangle \geqslant -\frac{\dot{v}(t)}{\rho(t)} ||x^a(t) - x^b(t)||^2$$

and hence

$$\frac{d}{dt}(||x^a(t) - x^b(t)||^2) \leqslant \frac{2\dot{v}(t)}{\rho(t)} ||x^a(t) - x^b(t)||^2.$$

Using the Gronwall inequality, we obtain for all $t \in [0, T]$

$$||x^a(t) - x^b(t)||^2 \leqslant ||a - b||^2 \exp\left(2 \int_0^t \frac{\dot{v}(s)}{\rho(s)} ds\right).$$

The proof is then complete. □

The following corollary is a direct consequence of the corollary above. The second case of convex sets is due to Moreau [23,26] for Hilbert spaces.

Corollary 2.2. *Assume that* $(\mathscr{H})$ *holds and* H *is finite dimensional. Then the assertions of Corollary* 2.1 *hold under any one of the following assumptions*:

(a) *there exists some number* $\rho > 0$ *such that all sets* $C(t)$ *are* ρ*-prox-regular and* T *is choosen such that* $2\int_0^T \dot{v}(s)\,ds < \rho$;
(b) *all the sets* $C(t)$ *are convex.*

Proof. In case (b), the sets $C(t)$ are ρ-prox-regular with $\rho = +\infty$. So, in both cases the function $t \mapsto \dot{v}(t)/\rho$ is integrable over $[0, T]$ and hence the result follows. □

3. Perturbation of sweeping process

Consider now a finite delay $r \geqslant 0$ and introduce in a classical way, for each $t \in [0, T]$ and each $x : [-r, T] \to H$, the mapping $x_t : [-r, 0] \to H$ with $x_t(s) = x(t + s)$. If $\mathscr{C}_T$ (resp. $\mathscr{C}_0$) denote the Banach space of continuous mappings from $[-r, T]$ (resp. $[-r, 0]$) into H, then the mapping $(t, x) \mapsto x_t$ is continuous over $[0, T] \times \mathscr{C}_T$.

Let Φ be a set-valued mapping from $[0,T]\times\mathscr{C}_0$ into the nonempty weakly compact convex subsets of H. We assume that Φ is scalarly $\mathscr{L}([0,T])\otimes\mathscr{B}(\mathscr{C}_0)$ measurable with respect to both variables and scalarly upper semicontinuous with respect to the second variable. Here $\mathscr{L}([0,T])$ (resp. $\mathscr{B}(\mathscr{C}_0)$) denotes the Lebesgue (resp. Borel)-measurable subsets of $[0,T]$ (resp. $\mathscr{C}_0$). We also assume that there exists an integrable function m on $[0,T]$ such that for all $u\in\mathscr{C}_0$

$$|\Phi(t,u)| := \sup\{\|e\| : e\in\Phi(t,u)\}\leqslant m(t).$$

We are interested, as in Castaing and Monteiro Marques [8], in the perturbed differential inclusion over $[0,T]$ given by

$$\text{(III)}\qquad \dot{x}(t)\in -N_{C(t)}(x(t))+\Phi(t,x_t)\quad\text{with } x_0=\varphi,$$

where $\varphi\in\mathscr{C}_0$ is fixed and satisfies $\varphi(0)\in C(0)$.

We still assume that the sets $C(t)$ are $\rho(t)$-*prox-regular* and we are going to show how the results in Section 2 can be applied to solve inclusion (III).

We denote by w the absolutely continuous function defined for each $t\in[0,T]$ by

$$w(t)=\int_0^t (\dot{v}(s)+m(s))\,ds,$$

and we assume $2\int_0^t w(s)\,ds<\rho(t)$ for $0<t<T$.

Theorem 3.1. *Assume that* $(\mathscr{H})$ *holds and* H *is separable. Then, a mapping* $x\in\mathscr{C}_T$ *that is absolutely continuous on* $[0,T]$, *is a solution of the constrained differential inclusion* (III) *iff it is a solution of the unconstrained differential inclusion*

$$\text{(IV)}\qquad \dot{x}(t)\in -\dot{w}(t)\partial d_{C(t)}(x(t))+\Phi(t,x_t)\quad \textit{with } x_0=\varphi.$$

Further, the set of solutions is nonempty whenever H *is finite dimensional.*

Proof. Suppose first that x is a solution of (III). Then, classical measurability techniques of measurable set-valued mappings give a measurable selection z of $t\mapsto\Phi(t,x_t)$ such that

$$\dot{x}(t)\in -N_{C(t)}(x(t))+z(t). \tag{3.1}$$

We use a classical idea (see for example [6]) by putting for all $t\in[0,T]$

$$\zeta(t)=\int_0^t z(s)\,ds,\quad D(t)=C(t)-\zeta(t)\quad\text{and}\quad y(t)=x(t)-\zeta(t).$$

It is easily verified that $D(t)$ is normally regular for all $t \in [0, T]$. Moreover for $s, t \in [0, T]$ and for $e \in H$ one has

$$\begin{aligned} |d(e, D(t)) - d(e, D(s))| &\leqslant |d(e + \zeta(t), C(t)) - d(e + \zeta(s), C(s))| \\ &\leqslant ||\zeta(t) - \zeta(s)|| + |d(e + \zeta(t), C(t)) \\ &\quad - d(e + \zeta(t), C(s))| \\ &\leqslant ||\zeta(t) - \zeta(s)|| + |v(t) - v(s)| \\ &\leqslant |w(t) - w(s)|. \end{aligned}$$

As inclusion (3.1) can be rewritten in the form $\dot{y}(t) \in -N_{D(t)}(y(t))$ and as $y(0) = x(0) = \varphi(0)$ and $\varphi(0) \in C(0) = D(0)$, it follows from Proposition 2.1 that for almost all $t \in [0, T]$

$$\dot{y}(t) \in -\dot{w}(t) \partial d_{D(t)}(y(t)).$$

This means that

$$\dot{x}(t) \in -\dot{w}(t) \partial d_{C(t)}(x(t)) + z(t),$$

which ensures

$$\dot{x}(t) \in -\dot{w}(t) \partial d_{C(t)}(x(t)) + \Phi(t, x_t),$$

and hence x is a solution of (IV).

Suppose now that x is a solution of the unconstrained differential inclusion (IV). As above, there exists a measurable selection z of the set-valued mapping $t \mapsto \Phi(t, x_t)$ such that for almost all $t \in [0, T]$

$$\dot{x}(t) \in -\dot{w}(t) \partial d_{C(t)}(x(t)) + z(t). \tag{3.2}$$

Define $\zeta(t)$, $D(t)$ and $y(t)$ for all $t \in [0, T]$ exactly as above. The sets $D(t)$ obviously inherit the $\rho(t)$-prox-regularity from the sets $C(t)$ and as above once again for all $s, t \in [0, T]$ and for all $e \in H$ we have

$$|d(e, D(t)) - d(e, D(s))| \leqslant |w(t) - w(s)|,$$

and by assumption we also have $2 \int_0^t \dot{w}(s)\, ds < \rho(t)$. So, observing that inclusion (3.2) is equivalent to

$$\dot{y}(t) \in -\dot{w}(t) \partial d_{D(t)}(y(t))$$

and that $y(0) \in D(0)$, we obtain by Theorem 2.1 that

$$\dot{y}(t) \in -N_{D(t)}(y(t)),$$

that is,

$$\dot{x}(t) \in -N_{C(t)}(x(t)) + z(t).$$

This ensures

$$\dot{x}(t) \in -N_{C(t)}(x(t)) + \Phi(t, x_t)$$

and hence x is a solution of the constrained differential inclusion (III).

It remains to show that, in the finite-dimensional setting, the unconstrained differential inclusion (IV) admits at least a solution $x \in \mathscr{C}_T$ that is absolutely continuous on $[0, T]$. Using the Clarke directional derivative of Lipschitz functions, one easily checks that the set-valued mapping with compact convex values given for $(t, e) \in [0, T] \times H$ by

$$(t, e) \mapsto -\dot{w}(t) \partial d_{C(t)}(e)$$

is $\mathscr{L}([0, T]) \otimes \mathscr{B}(H)$-measurable with respect to both variables and upper semicontinuous with respect to the second variable. Note also that for each $x \in \mathscr{C}_T$ and each $t \in [0, T]$ one has $x(t) = x_t(0)$. Then, putting for $(t, u) \in [0, T] \times \mathscr{C}_0$,

$$\Phi_0(t, u) = -\dot{w}(t) \partial d_{C(t)}(u(0)) + \Phi(t, u)$$

the differential inclusion (IV) appears in the form

$$\dot{x}(t) \in \Phi_0(t, x_t) \quad \text{with } x_0 = \varphi$$

as a classical unconstrained differential inclusion with delay. Indeed, the second member is given by a set-valued mapping with compact convex values that is $\mathscr{L}([0, T]) \otimes \mathscr{B}(\mathscr{C}_0)$-measurable and upper semicontinuous with respect to $u \in \mathscr{C}_0$ and for all $u \in \mathscr{C}_0$

$$\sup\{\|e\| : e \in \Phi_0(t, u)\} \leqslant \dot{w}(t) + m(t).$$

So, applying corresponding results in [1] one obtains that the differential inclusion (IV) admits at least one solution, and the proof is complete. $\square$

As any closed convex set is ∞-prox-regular, the following result is a direct consequence of Theorem 2.1. It has been established by Castaing and Monteiro Marques [8].

Corollary 3.1. *Assume that H is finite dimensional and that all the sets $C(t)$ are convex and satisfy $(\mathscr{H})$. Then, under the assumptions above, the constrained differential inclusion* (III) *admits at least one solution.*

A second corollary will be obtained by taking the delay $r = 0$. In order to state this corollary, consider a set-valued mapping Γ from $[0, T] \times H$ into the nonempty weakly compact subsets of H. Assume that Γ is scalarly $\mathscr{L}([0,T])\otimes\mathscr{B}(H)$-measurable with respect to both variables and scalarly upper semicontinuous with respect to the second variable. Assume also that there exists some integrable function m over $[0,T]$ such that for all $x\in H$

$$\sup\{||e|| : e\in\Gamma(t,x)\}\leqslant m(t).$$

Assume that the set $C(t)$ are $\rho(t)$-prox-regular with $\int_0^t \dot{w}(s)\,ds<\rho(t)$, where $\dot{w}(s) := |\dot{v}(s)| + m(s)$. Then, the following corollary can be provided as a direct consequence of Theorem 3.1 with $r=0$. The existence part in the convex case appeared in [6].

Corollary 3.2. *Assume that* $(\mathscr{H})$ *holds and H is separable. Then, under the assumptions above, an absolutely continuous mapping* $x:[0,T]\to H$ *is a solution of the constrained differential inclusion*

$$\dot{x}(t)\in -N_{C(t)}(x(t))+\Gamma(t,x(t)) \quad \text{with } x(0)=x_0\in C(0)$$

iff it is a solution of the unconstrained differential inclusion

$$\dot{x}(t)\in -\dot{w}(t)\partial d_{C(t)}(x(t))+\Gamma(t,x(t)) \quad \text{with } x(0)=x_0\in C(0).$$

Furthermore, the set of solutions is nonempty whenever H is finite dimensional.

Another corollary concerning topological properties of the set of solutions is obtained by applying corresponding results in [1,9] to the second differential inclusion in Corollary 3.2.

Corollary 3.3. *Assume that* $(\mathscr{H})$ *holds and H is finite dimensional and denote by* $\mathscr{S}_a$ *the set of solutions of the differential inclusion*

$$\dot{x}(t)\in -N_{C(t)}(x(t))+\Gamma(t,x(t))$$

associated with the initial value $a\in C(0)$. *Then, each* $\mathscr{S}_a$ (*for* $a\in C(0)$) *is a nonempty compact subset of* $\mathscr{C}([0,T],H)$ *endowed with the sup norm and the set-valued mapping* $a\mapsto\mathscr{S}_a$ *is upper semicontinuous over* $C(0)$.

Consider now the case when $C(t)$ does not depend on t. Let K be a nonempty closed subset of H that is ρ-prox-regular for some $\rho>0$. Assume that $2\int_0^t m(s)\,ds<\rho$ for $t<T$. Then, we have the following topological result. Note that the existence alone of a solution will be established in Corollary 4.1 without the prox-regularity assumption. The importance of this differential inclusion in economic mathematical problems as well as earlier results by B. Cornet and C. Henry will be recalled before the statement of Corollary 4.1.

Corollary 3.4. *Assume that H is finite dimensional. Then, for each $a \in K$ the set $\mathscr{S}_a$ of solutions of the differential inclusion*

$$\dot{x}(t) \in -N_K(x(t)) + \Gamma(t, x(t))$$

is a nonempty compact subset of $\mathscr{C}([0,T], H)$ and the set-valued mapping $a \mapsto \mathscr{S}_a$ is upper semicontinuous on K.

4. Sweeping process with nonregular sets

In this section, we drop the regularity assumption on the closed sets $C(t)$ and merely assume that they move in an absolutely continuous way, i.e., we suppose that $(\mathscr{H})$ holds. To get a solution of (I), we take advantage of the strong connection provided by Section 1 between the solutions of (I) and (II), at least in the regularity case. Indeed, Section 1 tells us that a natural candidate set-valued mapping with nonempty convex compact values $G : [0,T] \times H \rightrightarrows H$ whose associated differential inclusion would allow to get a solution of (I) with the general theory of differential inclusions with convex compact second member is given by

$$G(t,x) = -\dot{v}(t)\partial d_{C(t)}(x).$$

So, in absence of any regularity, we are naturally led to consider the differential inclusion (II) on which we impose the constraints

$$x(t) \in C(t) \quad \text{for all } t \in [0,T].$$

This will be done by applying viability theory results to this constrained differential inclusion. We also refer to Castaing [4] and Valadier [34] for the use of some viability results to the study of some sweeping process.

We begin by recalling some notions and results about the viability for differential inclusions defined by set-valued mappings with nonempty convex compact values.

Let $G : [0,T] \times H \rightrightarrows H$ be a set-valued mapping with nonempty convex compact values that is $\mathscr{L}([0,T]) \otimes \mathscr{B}(H)$-measurable and upper semicontinuous with respect to $x \in H$ for almost all $t \in [0,T]$. Assume also that the standard growth condition holds, i.e., there exist an integrable function γ over $[0,T]$ such that for almost all $t \in [0,T]$ and all $x \in H$

$$|G(t,x)| \leqslant \gamma(t)(1 + \|x\|).$$

Let also for each $t \in [0,T]$ a nonempty closed subset $S(t)$ of H. Assume also that S is absolutely continuous on $[0,T]$ (see, e.g., [16,17,28]) in the sense that for all real numbers $\varepsilon > 0$ and $\rho > 0$ there exists a real number $\delta > 0$ such that for all $0 \leqslant s_1 < t_1 \leqslant \cdots s_m < t_m \leqslant T$ satisfying $\sum_{i=1}^{m} (t_i - s_i) \leqslant \delta$ one has

$$\sum_{i=1}^{m} \max(e_\rho(S(s_i), S(t_i))), e_\rho(S(t_i), S(s_i)) \leqslant \varepsilon,$$

where, for two closed subsets P and Q in H,

$$e_\rho(P,Q) = \inf\{\eta > 0 : P \cap \rho\mathbb{B}_H \subset Q + \eta\mathbb{B}\}.$$

Denote by $h_G(t,x,y)$ the lower hamiltonian function of G defined on $[0,T] \times H \times H$ by

$$h_G(t,x,y) = \min\{\langle y,z\rangle : z \in G(t,x)\}.$$

The following viability result has been proved by Frankowska and Plaskacz [16], see Theorem 3.1 and Corollary 3.2 therein. Recall that the graph of S is defined by

$$\operatorname{gph} S = \{(t,x) : x \in S(t)\}.$$

Theorem 4.1 (Frankowska and Plaskacz [16]). *Suppose in addition to the assumptions above that H is finite dimensional. Then, the following assertions are equivalent*:

(a) *there exists a set $L \subset [0,T]$ of full measure such that for all $t \in L$ and all $x \in S(t)$ one has*

$$\alpha + h_G(t,x,y) \leqslant 0 \quad \textit{for all } (\alpha,y) \in N^F(\operatorname{gph} S; t,x);$$

(b) *for every $t_0 \in [0,T[$ and every $u_0 \in S(t_0)$ there exists an absolutely continuous solution $x(.)$, over the interval $[t_0,T]$, of the differential inclusion*

$$\dot{x}(t) \in G(t,x(t)) \quad \textit{with } x(t_0) = u_0$$

satisfying the constraints

$$x(t) \in S(t) \quad \textit{for all } t \in [t_0,T].$$

The following result will also be needed later and it has its own interest. The second equality that it provides describes the Fréchet normal cone to the graph of a set valued mapping M in terms of the function Δ_M given by

$$\Delta_M(x,y) := d(y, M(x)),$$

with the convention $d(y,\phi) = +\infty$. A similar result has been established in Thibault [31] for the limiting Fréchet normal cone with the use of the Ekeland variational principle. As it appears below, the proof is much more simple for the Fréchet normal cone and the first equality is more precise.

Proposition 4.1. *Let M be a set-valued mapping with closed graph between two normed vector spaces X and Y and let $(\bar{x},\bar{y}) \in \operatorname{gph} M$. Then*

$$\partial^F \Delta_M(\bar{x},\bar{y}) = N^F(\operatorname{gph} M; \bar{x},\bar{y}) \cap (X^* \times \mathbb{B}_{Y^*})$$

and

$$N^F(\mathrm{gph}\, M; \bar{x}, \bar{y}) = \mathbb{R}_+ \partial^F \Delta_M(\bar{x}, \bar{y}).$$

Proof. The second equality is obviously a direct consequence of the first one. So, we begin by proving the inclusion $\partial^F \Delta_M(\bar{x}, \bar{y}) \subset N^F(\mathrm{gph}\, M; \bar{x}, \bar{y}) \cap (X^* \times \mathbb{B}_Y)$. Fix any (x^*, y^*) in $\partial^F \Delta_M(\bar{x}, \bar{y})$ and fix $\varepsilon > 0$. There exists some real number $\eta > 0$ such that for all $x \in X$ with $\|x - \bar{x}\| \leqslant \eta$ and all $y \in Y$ with $\|y - \bar{y}\| \leqslant \eta$ one has

$$\begin{aligned}&\langle x^*, x - \bar{x} \rangle + \langle y^*, y - \bar{y} \rangle \\ &\quad \leqslant d(y, M(x)) - d(\bar{y}, M(\bar{x})) + \varepsilon(\|x - \bar{x}\| + \|y - \bar{y}\|). \qquad (4.1)\end{aligned}$$

So, for any $(x, y) \in \mathrm{gph}\, M$ with $\|x - \bar{x}\| \leqslant \eta$ and $\|y - \bar{y}\| \leqslant \eta$, we obtain (because $d(\bar{y}, M(\bar{x})) = 0$)

$$\langle x^*, x - \bar{x} \rangle + \langle y^*, y - \bar{y} \rangle \leqslant \varepsilon(\|x - \bar{x}\| + \|y - \bar{y}\|).$$

This implies $(x^*, y^*) \in N^F(\mathrm{gph}\, M; \bar{x}, \bar{y})$. Further, taking $x = \bar{x}$ in (4.1) one easily obtains that $\|y^*\| \leqslant 1$, and the inclusion

$$\partial^F \Delta_M(\bar{x}, \bar{y}) \subset N^F(\mathrm{gph}\, M; \bar{x}, \bar{y}) \cap (X^* \cap \mathbb{B}_Y) \qquad (4.2)$$

follows.

Consider now the converse inclusion. Fix any (x^*, y^*) in $N^F(\mathrm{gph}\, M; \bar{x}, \bar{y})$ with $\|y^*\| < 1$ and fix any $\varepsilon > 0$ satisfying $\varepsilon + \|y^*\| < 1$. There exists, by the definition of Fréchet normal cone, some real number $\eta' > 0$ such that for all $(x', y') \in \mathrm{gph}\, M$ with $\|x' - \bar{x}\| \leqslant 3\eta'$ and $\|y' - \bar{y}\| \leqslant 3\eta'$ one has

$$\langle x^*, x' - \bar{x} \rangle + \langle y^*, y' - \bar{y} \rangle \leqslant \varepsilon(\|x' - \bar{x}\| + \|y' - \bar{y}\|). \qquad (4.3)$$

Fix a positive number η satisfying $(1 + \|x^*\|)\eta < \eta'$ and fix $x \in X$ and $y \in Y$ with $\|x - \bar{x}\| \leqslant \eta$ and $\|y - \bar{y}\| \leqslant \eta$. If $d(y, M(x)) > \eta'$, then

$$\langle x^*, x - \bar{x} \rangle + \langle y^*, y - \bar{y} \rangle \leqslant (\|x^*\| + 1)\eta < d(y, M(x))$$

and hence

$$\begin{aligned}&\langle x^*, x - \bar{x} \rangle + \langle y^*, y - \bar{y} \rangle \\ &\quad \leqslant d(y, M(x)) - d(\bar{y}, M(\bar{x})) + \varepsilon(\|x - \bar{x}\| + \|y - \bar{y}\|). \qquad (4.4)\end{aligned}$$

Suppose now that $d(y, M(x)) \leqslant \eta'$. Choose $y' \in M(x)$ such that

$$\max(1/2, \varepsilon + ||y^*||)||y - y'|| \leqslant d(y, M(x)) \tag{4.5}$$

and observe that

$$||y' - \bar{y}|| \leqslant ||y - \bar{y}|| + ||y - y'|| \leqslant \eta + 2\eta' < 3\eta'.$$

So, we obtain from (4.3) that

$$\begin{aligned} \langle x^*, x - \bar{x} \rangle + \langle y^*, y - \bar{y} \rangle &= \langle x^*, x - \bar{x} \rangle + \langle y^*, y' - \bar{y} \rangle + \langle y^*, y - y' \rangle \\ &\leqslant \varepsilon(||x - \bar{x}|| + ||y' - \bar{y}||) + ||y^*||.||y - y'|| \\ &\leqslant \varepsilon(||x - \bar{x}|| + ||y - \bar{y}||) + (\varepsilon + ||y^*||)||y - y'||. \end{aligned}$$

It then follows from (4.5) that

$$\langle x^*, x - \bar{x} \rangle + \langle y^*, y - \bar{y} \rangle \leqslant \varepsilon(||x - \bar{x}|| + ||y - \bar{y}||) + d(y, M(x)) - d(\bar{y}, M(\bar{x})).$$

This inequality and (4.4) ensure that $(x^*, y^*) \in \partial^F \Delta_M(\bar{x}, \bar{y})$. As the Fréchet subdifferential of a function is always strongly closed (see for example [3]) we conclude that

$$(x^*, y^*) \in \partial^F \Delta_M(\bar{x}, \bar{y})$$

for any $(x^*, y^*) \in N^F(\text{gph}\, M; \bar{x}, \bar{y})$ satisfying $||y^*|| \leqslant 1$. This means that the converse inclusion of (4.2) also holds, and hence the proof is complete. $\square$

We are now ready to apply Theorem 4.1 to the differential inclusion (II) subject to the constraints $x(t) \in C(t)$ for all $t \in [0, T]$.

Proposition 4.2. *Assume that H is finite dimensional and the closed sets $C(t)$ merely satisfy the hypothesis $(\mathscr{H})$. Then, there exists an absolutely continuous solution $x(.)$ of the differential inclusion*

$$\dot{x}(t) \in -\dot{v}(t)\partial d_{C(t)}(x(t)) \quad \textit{with } x(0) = x_0 \in C(0)$$

satisfying the constraints

$$x(t) \in C(t) \quad \textit{for all } t \in [0, T].$$

Proof. For $(t, x) \in [0, T] \times H$ put

$$G(t, x) := -\dot{v}(t)\partial d_{C(t)}(x).$$

It is not difficult to verify that G satisfies the measurability assumption and the growth condition in the beginning of Section 4. Moreover, it is also easily seen that hypothesis $(\mathscr{H})$ ensures that C is absolutely continuous on $[0, T]$ in the sense recalled above. Let us now prove that condition (a) in Theorem 4.1 also holds. Fix any point $\bar{r} \in]0, T[$ where the function v is derivable and take $\bar{x} \in C(\bar{r})$. Fix also any $(\alpha, \xi) \in \partial^F \Delta_C(\bar{r}, \bar{x})$. For each real number $\varepsilon > 0$, there exists, by definition of Fréchet subgradients, some real number $\eta > 0$ such that for all $r \in]0, T[$ with $|r - \bar{r}| \leqslant \eta$ and all $x \in H$ with $||x - \bar{x}|| \leqslant \eta$ one has

$$\langle (\alpha, \xi); (r - \bar{r}, x - \bar{x}) \rangle \leqslant \Delta_C(r, x) + \varepsilon(|r - \bar{r}| + ||x - \bar{x}||),$$

that is,

$$\alpha(r - \bar{r}) + \langle \xi, x - \bar{x} \rangle \leqslant d(x, C(r)) + \varepsilon(|r - \bar{r}| + ||x - \bar{x}||). \tag{4.6}$$

Taking $r = \bar{r}$ in (4.6) we obtain for all $x \in H$ with $||x - \bar{x}|| \leqslant \eta$

$$\langle \xi, x - \bar{x} \rangle \leqslant d(x, C(\bar{r})) + \varepsilon ||x - \bar{x}||$$

and hence (since $d(\bar{x}, C(\bar{r})) = 0$)

$$\langle \xi, x - \bar{x} \rangle \leqslant d(x, C(\bar{r})) - d(\bar{x}, C(\bar{r})) + \varepsilon ||x - \bar{x}||.$$

This means that

$$\xi \in \partial^F d_{C(\bar{r})}(\bar{x}). \tag{4.7}$$

By assumption $(\mathscr{H})$, for any $r \in]0, T[$ there exists some $x_r \in C(r)$ with

$$||x_r - \bar{x}|| \leqslant |v(r) - v(\bar{r})|.$$

Taking $x = x_r$ as above in (4.6) we get for all r sufficiently close to $\bar{r}$

$$\begin{aligned} \alpha(r - \bar{r}) &\leqslant \langle -\xi, x_r - \bar{x} \rangle + \varepsilon(|r - \bar{r}| + ||x_r - \bar{x}||) \\ &\leqslant ||\xi||.|v(r) - v(\bar{r})| + \varepsilon(|r - \bar{r}| + |v(r) - v(\bar{r})|) \end{aligned}$$

and hence

$$|\alpha| \leqslant \dot{v}(\bar{r}) ||\xi||. \tag{4.8}$$

If $\xi = 0$, then (4.8) entails that $\alpha = 0$ and hence in this case

$$\alpha + h_G(\bar{r}, \bar{x}, \xi) = 0.$$

Suppose that $\xi \neq 0$. Then (4.7) implies that $\frac{\xi}{||\xi||} \in \partial^F d_{C(\bar{r})}(\bar{x}) \subset \partial d_{C(\bar{r})}(\bar{x})$ and hence

$$-\dot{v}(\bar{r}) \frac{\xi}{||\xi||} \in G(\bar{r}, \bar{x}).$$

Furthermore, one has by (4.8)

$$\alpha + \left\langle -\dot{v}(\bar{r})\frac{\xi}{||\xi||}, \xi \right\rangle = \alpha - \dot{v}(\bar{r})||\xi|| \leqslant 0.$$

So, in both cases

$$\alpha + h_G(\bar{r}, \bar{x}, \xi) \leqslant 0,$$

and, by Proposition 4.1, this inequality still holds for all $(\alpha, \xi) \in N^F(\mathrm{gph}\, C; \bar{r}, \bar{x})$. Consequently, we may apply Theorem 4.1 to get that the set-valued mapping C is viable for G, that is, there exists an absolutely continuous solution of the differential inclusion

$$\dot{x}(t) \in G(t, x(t)) \quad \text{with } x(0) = x_0$$

satisfying the constraints

$$x(t) \in C(t) \quad \text{for all } t \in [0, T].$$

The proof is then complete. □

The proposition above allows us to provide the general theorem below concerning the existence of solution for sweeping process with general closed sets.

Theorem 4.2. *Assume that H is finite dimensional and the closed sets $C(t)$ merely satisfy the hypothesis $(\mathscr{H})$. Then the differential inclusion* (I) *admits at least one absolutely continuous solution $x(.)$ satisfying for almost all $t \in [0, T]$*

$$||\dot{x}(t)|| \leqslant \dot{v}(t).$$

Proof. The theorem is a direct consequence of Proposition 4.2 and the fact that $||y|| \leqslant 1$ for any $y \in \partial d_Q(x)$, any subset Q of H, and any $x \in H$. □

We can also consider, in the case of general closed sets $C(t)$, a special perturbation with a growth condition depending on the state variable. For a subset $S \subset H$ we denote by $T(S; \bar{x})$ the Bouligand contingent cone to S at $\bar{x} \in S$ (see, e.g., [11]). It is not difficult to see that

$$\langle u^*, u \rangle \leqslant 0 \quad \text{for all } u \in T(S; \bar{x}) \quad \text{and} \quad u^* \in N^F(S; \bar{x}). \tag{4.9}$$

As it appears below, the perturbation with which we are able to deal requires a partial tangential assumption with respect to the state variable using the contingent cone.

Theorem 4.3. *Assume that H is finite dimensional and the closed sets $C(t)$ merely satisfy $(\mathscr{H})$. Let $\Gamma : [0,T] \times H \to H$ be a set-valued mapping with nonempty compact convex values that is measurable with respect to (t,x) and upper semicontinuous with respect to x. Assume that for some set $L' \subset [0,T]$ of full measure, some integrable function σ and some real number $\beta \geqslant 0$ one has for all $t \in L'$ and $x \in C(t)$*

$$\Gamma(t,x) \cap T(C(t);x) \neq \emptyset \quad \textit{and} \quad |\Gamma(t,x)| \leqslant \sigma(t)(1+\beta||x||).$$

Then, the perturbed sweeping process

$$\dot{x}(t) \in -N_{C(t)}(x(t)) + \Gamma(t,x(t)) \quad \textit{with } x(0) = x_0 \in C(0)$$

admits at least one absolutely continuous solution $x(.)$ satisfying

$$||\dot{x}(t)|| \leqslant \dot{v}(t) + \sigma(t)(1+\beta||x(t)||) \quad \textit{for almost all } t \in [0,T].$$

Proof. Putting

$$G(t,x) := -\dot{v}(t)\partial d_{C(t)}(x),$$

we obtain as in the proof of Proposition 4.2, for any fixed $(\bar{r},\bar{x}) \in \operatorname{gph} C$ with $\bar{r} \in]0,T[\cap L'$ and v derivable at $\bar{r}$, that for all $(\alpha,\xi) \in N^F(\operatorname{gph} C;\bar{r},\bar{x})$ one has

$$\alpha + h_G(\bar{r},\bar{x},\xi) \leqslant 0.$$

Our tangential assumption on Γ and (4.9) ensure that for the lower hamiltonian h_Γ of Γ, we have $h_\Gamma(\bar{r},\bar{x},\xi) \leqslant 0$. As it is easily seen that $h_{G+\Gamma} = h_G + h_\Gamma$, we get

$$\alpha + h_{G+\Gamma}(\bar{r},\bar{x},\xi) \leqslant 0 \quad \text{for all } (\alpha,\xi) \in N^F(\operatorname{gph} C;\bar{r},\bar{x}).$$

So, we may apply Theorem 4.1 to obtain a solution $x(.)$ of the differential inclusion

$$\dot{x}(t) \in -\dot{v}(t)\partial d_{C(t)}(x(t)) + \Gamma(t,x(t)) \quad \text{with } x(0) = x_0 \in C(0) \tag{4.10}$$

satisfying the constraints

$$x(t) \in C(t) \quad \text{for all } t \in [0,T].$$

The inclusion (4.10) implies, on the one hand, that for almost all $t \in [0,T]$

$$\dot{x}(t) \in -N_{C(t)}(x(t)) + \Gamma(t,x(t))$$

because $\partial d_S(u) \subset N_S(u)$ for any $u \in S$. On the other hand, the growth condition on Γ and (4.10) imply that almost everywhere

$$||\dot{x}(t)|| \leqslant \dot{v}(t) + \sigma(t)(1+\beta||x(t)||).$$

The proof is then complete. $\square$

When the growth condition upon the set-valued perturbation Γ does not depend on the state variable, a more general result can be established.

Theorem 4.4. *Assume that H is finite dimensional and the closed sets $C(t)$ merely satisfy $(\mathscr{H})$. Let $\Gamma : [0, T] \times H \to H$ be a set-valued mapping with nonempty compact convex values that is measurable with respect to (t, x) and upper semicontinuous with respect to x. Assume that there exists an integrable function m over $[0, T]$ such that for all $t \in [0, T]$ and all $x \in C(t)$*

$$\min\{||e|| : e \in \Gamma(t, x)\} \leqslant m(t). \tag{4.11}$$

Then the perturbed sweeping process

$$\dot{x}(t) \in -N_{C(t)}(x(t)) + \Gamma(t, x(t)) \quad \textit{with } x(0) = x_0 \in C(0)$$

admits at least one absolutely continuous solution $x(.)$. Further, this solution also satisfies

$$||\dot{x}(t)|| \leqslant \dot{v}(t) + 2m(t) \quad \textit{for almost all } t \in [0, T], \tag{4.12}$$

whenever, instead of (4.11) *we assume that for all $t \in [0, T]$ and all $x \in C(t)$*

$$|\Gamma(t, x)| \leqslant m(t). \tag{4.13}$$

Proof. Assume that (4.11) holds. Put

$$\Phi(t, x) := -(\dot{v}(t) + m(t))\partial_{C(t)}(x) + \Gamma(t, x)$$

and fix any $\bar{r} \in]0, T[$ where the function v is derivable and take any $\bar{x} \in C(\bar{r})$. As in the first part of the proof of Proposition 4.2, we obtain for any fixed $(\alpha, \xi) \in \partial^F \Delta_C(\bar{r}, \bar{x})$ that

$$|\alpha| \leqslant \dot{v}(\bar{r}).||\xi|| \quad \text{and} \quad \xi \in \partial^F d_{C(\bar{r})}(\bar{x}). \tag{4.14}$$

By our assumption on Γ, we may choose some $\bar{y} \in \Gamma(\bar{r}, \bar{x})$ with $||\bar{y}|| \leqslant m(\bar{r})$. If $\xi \neq 0$, then the second relation in (4.14) ensures that

$$-(\dot{v}(\bar{r}) + m(\bar{r}))\frac{\xi}{||\xi||} + \bar{y} \in \Phi(\bar{r}, \bar{x}),$$

and it follows from the inequality in (4.14) that

$$\alpha + \left\langle -(\dot{v}(\bar{r}) + m(\bar{r}))\frac{\xi}{||\xi||} + \bar{y}, \xi \right\rangle \leqslant (\alpha - \dot{v}(\bar{r})||\xi||) + ||\xi||(||\bar{y}|| - m(\bar{r})) \leqslant 0.$$

Taking the definition of the lower hamiltonian into account, we obtain

$$\alpha + h_{\Phi}(\bar{r}, \bar{x}, \xi) \leqslant 0.$$

When $\xi = 0$, the inequality in (4.14) entails that $\alpha = 0$. So, the last inequality above still holds even for $\xi = 0$. Proposition 4.1 says that this inequality also holds for all $(\alpha, \xi) \in N^F(\operatorname{gph} C; \bar{r}, \bar{x})$. Then, it follows from Theorem 4.1 that the set-valued mapping C is viable for ϕ, which means that there exists an absolutely continuous mapping $x(.)$ such that $x(t) \in C(t)$ for all $t \in [0, T]$ and

$$\dot{x}(t) \in \Phi(t, x(t)) \quad \text{a.e.} \quad \text{and} \quad x(0) = x_0 \in C(0).$$

The mapping $x(.)$ is easily seen to be a solution of the perturbed sweeping process.

If (4.13) is assumed in place of (4.11), using the definition of Φ one sees that the solution $x(.)$ above also satisfies inequality (4.12). The proof is then complete. □

As a direct corollary we obtain the following result where the sets $C(t)$ do not depend on t, i.e., $C(t) = K$ for all $t \in [0, T]$. The case when $\Gamma(t, x)$ does not depend on t either has been extensively studied. The problem has been first introduced and solved by Henry [19] with the convexity assumption on the set K. The convexity assumption has been relaxed by Cornet [14] who merely required the tangential regularity (or equivalently the normal regularity). Other cases have been studied by Malivert [20]. This differential inclusion is crucial in the study of planning procedures problems, see [14,19]. Here, we derive an existence of viable solution for any closed set K. Further, the set-valued mapping Γ may be time-dependent.

Corollary 4.1. *Assume that H is finite dimensional and that Γ satisfies* (4.13) *in Theorem* 4.4. *Let K be a closed subset of H with $x_0 \in K$. Then, the perturbed process*

$$\dot{x}(t) \in -N_K(x(t)) + \Gamma(t, x(t)) \quad \textit{with } x(0) = x_0$$

admits at least one absolutely continuous solution $x(.)$ satisfying

$$||\dot{x}(t)|| \leqslant 2m(t) \quad \textit{for almost all } t \in [0, T].$$

When the upper bound (4.12) is not required and all the sets $C(t)$ are bounded, an existence result for the perturbed sweeping process still holds under growth conditions more general than (4.13).

Corollary 4.2. *Assume that H is finite dimensional, $C(0)$ is bounded, and the sets $C(t)$ merely satisfy $(\mathscr{H})$. Let $\Gamma : [0, T] \times H \to H$ be a set-valued mapping with nonempty compact convex values that is measurable with respect to (t, x) and upper semicontinuous with respect to x. Assume that there exists some integrable* *function σ such that for all $t \in [0, T]$ and all $x \in C(t)$*

$$\min\{||e|| : e \in \Gamma(t, x)\} \leqslant \sigma(t)(1 + ||x||).$$

Then the perturbed sweeping process

$$\dot{x}(t) \in -N_{C(t)}(x(t)) + \Gamma(t, x(t)) \quad \text{with } x(0) = x_0 \in C(0)$$

admits at least one absolutely continuous solution $x(.)$.

Proof. It is not difficult to see that assumption $(\mathscr{H})$ and the boundedness of $C(0)$ ensures that, for some $\beta \geqslant 0$, one has $||x|| \leqslant \beta$ for all $t \in [0, T]$ and all $x \in C(t)$. So, for $m(t) := \sigma(t)(1 + \beta)$, the function m is integrable over $[0, T]$ and satisfies (4.11). The corollary then follows from Theorem 4.4. $\square$

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